Trigonometry (3/5): Geometry of Triangles

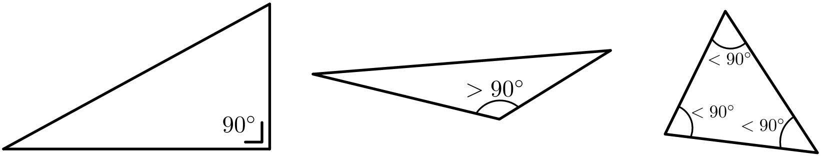
Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

Overview

1. Trigonometry in **right-angled** triangles
2. Trigonometry in **arbitrary** triangles
   - Law of sines
   - Law of cosines
   - Law of tangents
3. Formulas for area and perimeter
4. Height and distance problems

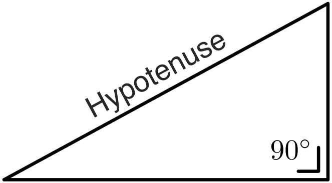
Terminology



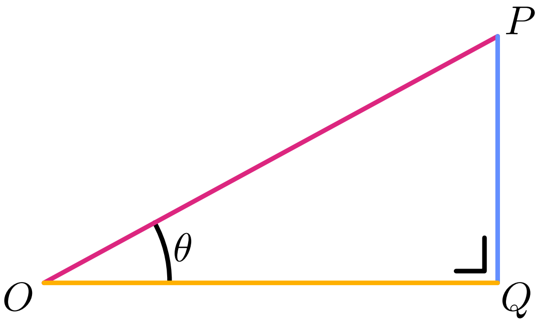
Right-angled   Obtuse   Acute

Different kinds of triangles:

- **Right-angled**: one angle exactly 90°
- **Obtuse**: one angle greater than 90°
- **Acute**: all angles less than 90°



Trigonometry in **right-angled** triangles



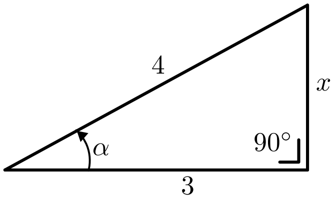
$$\sin\theta = \frac{PQ}{OP}$$

$$\cos\theta = \frac{OQ}{OP}$$

$$\tan\theta = \frac{PQ}{OQ}$$

Example

Find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$.



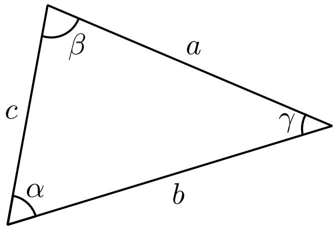
Example

A student sees the top of the Posco tower in central Songdo under an angle of 30°. Knowing that the Posco tower is approximately 300m tall, how far away is the student from the base of the tower?

Trigonometry in general triangles: law of sines

Formulas:

$$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}}$$



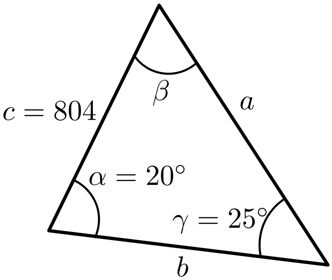
Useful when you know

- 2 angles + 1 side, or
- 1 angle + 2 sides

and want to know the others.

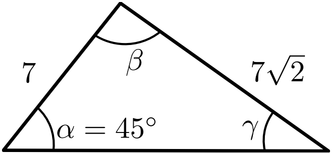
Example

Find a and b.



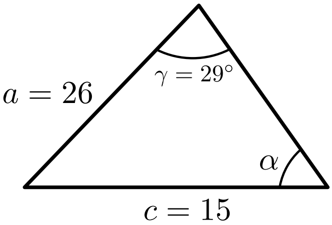
Ambiguous cases (1/3)

Find the angle γ.



Ambiguous cases (2/3)

Find the angle α.



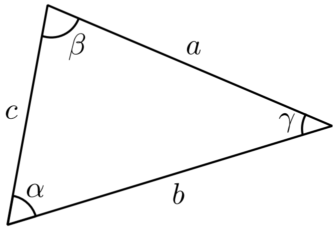
Ambiguous cases (3/3)

Given a triangle with angle $\alpha = 42^\circ$ and sides $a = 70$ and $b = 112$. Find the angle β.

Trigonometry in general triangles: law of cosines

Formulas:

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}}$$



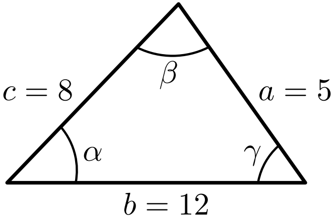
Useful when you know

- 2 sides + 1 angle in between, or
- 3 sides

and want to know the other side/angles.

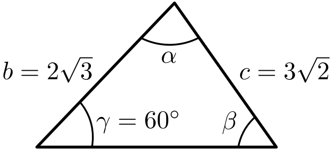
Example

Find the angles α, β, and γ.



Example

Find the angle α.



Example

If the ratio of the sides of a triangle is $a : b : c = 4 : 5 : 6$, prove that the greatest angle is twice the smallest angle.

Semi-perimeter formulas

- Express sin/cos as a function of the sides + semi-perimeter.
- Semi-perimeter: half ("semi") of the circumference ("perimeter")
- **You don't have to memorize these formulas, but you should know they exist.**

$$\boxed{\begin{aligned} \sin\frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin\frac{\beta}{2} &= \sqrt{\frac{(s-a)(s-c)}{ac}} \\ \sin\frac{\gamma}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned}}$$

$$\boxed{\begin{aligned} \cos\frac{\alpha}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \cos\frac{\beta}{2} &= \sqrt{\frac{s(s-b)}{ac}} \\ \cos\frac{\gamma}{2} &= \sqrt{\frac{s(s-c)}{ab}} \end{aligned}}$$

Formula for the area: Heron's formula

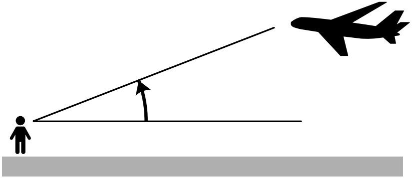
- Expresses area as a function of the lengths of the sides

$$\boxed{\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}}$$

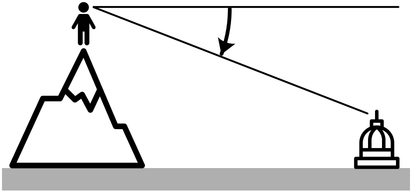
- Expresses sine of angles as function of area:

$$\boxed{\sin\alpha = 2\frac{\text{Area}}{bc}, \quad \sin\beta = 2\frac{\text{Area}}{ac}, \quad \sin\gamma = 2\frac{\text{Area}}{ab}.}$$

Problems involving height/distance: terminology



Angle of **elevation**: you look **up** at something



Angle of **depression**: you look **down** at something

Example

From a plane flying horizontally over a straight road, you see two road signs under an angle of 45° and 60°, respectively. The two road signs are 1 km apart. Find the height at which the plane is flying.

Problems for you to try (solution next lecture)

- You see a town on a hillside at an angle of elevation of 30°. You walk 80 meters (horizontally, along the ground) and see the town at an angle of elevation of 60°. Find the height of the town above ground level.
- A man lies on the ground and observes that a temple and a flagpole on that temple subtend equal angles at his eyes. If the height of the temple is 10m and that of the flagpole is 20m, find the subtended angles and the distance between the temple and the man.
- You are standing on the fortress walls, overlooking an approaching zombie army. You observe a zombie under an angle of depression of 45° and shoot an arrow. One second later, you shoot another arrow at the same zombie under an angle of depression of 60°. How soon will the zombie reach the base of the wall?