

Limits and Continuity (1/2)

Introduction to Engineering Mathematics

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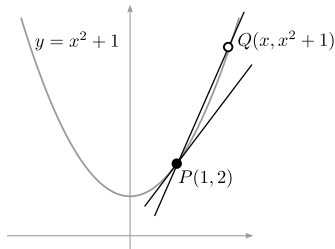
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Motivation

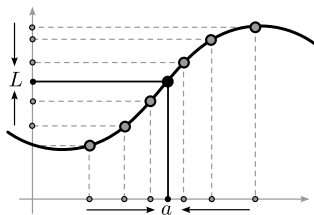
Compute the **tangent line** to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

- Take another point Q on the parabola
- Compute the line through P and Q
- Let Q “move towards” P

What is the slope of the line when Q gets “infinitely close” to P ?



Definition of limit



- We write:

$$\lim_{x \rightarrow a} f(x) = L$$

- We say: “The limit of $f(x)$ as x goes to a is L ”.
- We mean:
 - $f(x)$ is defined for all x near a (possibly not a itself)
 - As x gets closer to a , $f(x)$ gets closer and closer to L

Examples

Compute $\lim_{x \rightarrow 2} f(x)$, with $f(x)$ given by

- $f(x) = x - 1$ when $x \neq 2$
- $f(x) = 2$ otherwise.

Examples

$$\textcircled{1} \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

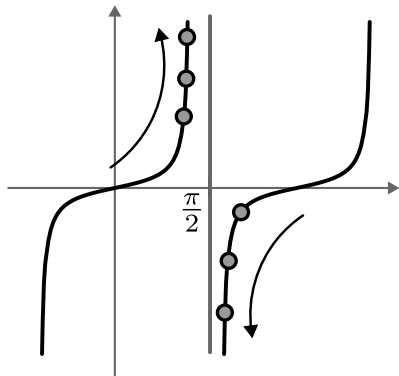
$$\textcircled{2} \lim_{x \rightarrow 0} \frac{x}{\sqrt{3+x} - \sqrt{3-x}}$$

One-sided limits

Sometimes the limit exists on “one side” of the function.

$$\lim_{x \rightarrow \pi/2} \tan x = ?$$

- $+\infty$ when $x \rightarrow 0$ from right
- $-\infty$ when $x \rightarrow 0$ from left

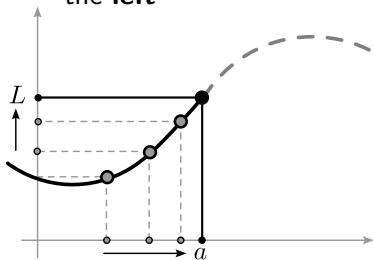


Definition

Left Limit:

$$\lim_{x \rightarrow a^-} f(x) = L$$

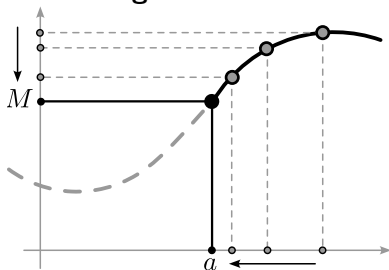
- $f(x)$ defined to the **left** of a
- $f(x) \rightarrow L$ as $x \rightarrow a$ from the **left**



Right Limit:

$$\lim_{x \rightarrow a^+} f(x) = M$$

- $f(x)$ defined to the **right** of a
- $f(x) \rightarrow M$ as $x \rightarrow a$ from the **right**



Relation with two-sided limit

The two-sided limit exists,

$$\lim_{x \rightarrow a} f(x) = L,$$

if and only if both one-sided limits exist and are equal:

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = L$$

Examples

$$\textcircled{1} \lim_{x \rightarrow 0^+} \sqrt{x}$$

$$\textcircled{2} \lim_{x \rightarrow 0} \frac{|x|}{x}$$

$$\textcircled{3} \lim_{x \rightarrow 2} \frac{|x - 2|}{x^2 + x - 6}$$

$$\textcircled{4} \lim_{x \rightarrow \pi} f(x), \text{ where } f(x) = \sin x \text{ when } x < \pi \text{ and } f(x) = \sqrt{x - \pi} \text{ otherwise.}$$

Limit laws

- 1 $\lim_{x \rightarrow a} x = a$
- 2 $\lim_{x \rightarrow a} c = c$
- 3 $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- 4 $\lim_{x \rightarrow a} (c(f(x))) = c \lim_{x \rightarrow a} f(x)$
- 5 $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- 6 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
- 7 $\lim_{x \rightarrow a} (f(x)^{m/n}) = (\lim_{x \rightarrow a} f(x))^{m/n} = L^{m/n}$,
 - If n even: L must be positive
 - If $m < 0$: L must be different from 0.

Limit laws

- ⑧ If $f(x) = g(x)$ for all $x \neq a$ close to a , then
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x).$$
- ⑨ If $f(x) \leq g(x)$ for all $x \neq a$ close to a , then
$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

These laws also hold for one-sided limits, with appropriate modifications.

Limits of polynomials and rational functions

- Polynomial: $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$$\lim_{x \rightarrow a} P(x) = P(a).$$

- Rational function: $F(x) = \frac{P(x)}{Q(x)}$, with $P(x)$, $Q(x)$ polynomials

$$\lim_{x \rightarrow a} F(x) = \frac{\lim_{x \rightarrow a} P(x)}{\lim_{x \rightarrow a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0.$$

The squeeze theorem

Lets you compute the limit of a difficult function $g(x)$ “squeezed” between two simple functions $f(x)$ and $h(x)$.

Suppose

- 1 $f(x) \leq g(x) \leq h(x)$ for some x near a
- 2 $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$.

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

Example

Compute $\lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right)$.

plot $x^2 \sin(1/x)$



NATURAL LANGUAGE



MATH INPUT



EXTENDED KEYBOARD



EXAMPLES



UPLOAD

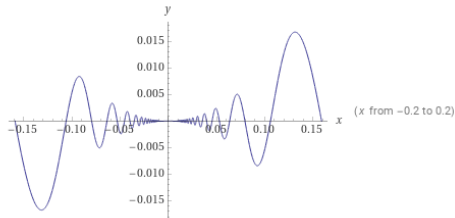


RANDOM

Input interpretation

plot $x^2 \sin\left(\frac{1}{x}\right)$

Plots



Source: Wolfram Alpha