## Derivatives (1/2)Introduction to Engineering Mathematics

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#### Motivation: tangent lines

In the previous class, we managed to compute the slope of the tangent line to the parabola  $y = x^2$  by computing the limit

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

How would we do this for an **arbitrary curve** y = f(x)?



#### Derivative of a function

We define the **derivative of** f(x) at x = a ("f prime") as

$$f'(a) = \lim_{x \to a} rac{f(x) - f(a)}{x - a}$$
 (if the limit exists).

If we put h = x - a, we can rewrite this as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is sometimes easier to compute.

If f'(a) exists, we say that f(x) is **differentiable** at x = a.

### Examples

Compute the derivative of  $f(x) = 3x^2 + 7x - 5$  at x = 1.

#### Examples

Compute the derivative of f(x) = |x| at x = a.

#### The derivative as a function

By letting a vary in f'(a), we obtain a function given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
 (if the limit exists).

There are many notations for the derivative function: for y = f(x),

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x).$$

all mean the same thing.

Example

Show that 
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$
.

### Properties of differentiable functions

We say that y = f(x) is differentiable on an interval [a, b] if

- f'(x) exists for each  $x \in (a, b)$
- At x = a, the *right derivative* exists:

$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}.$$

• At x = b, the *left derivative* exists:

$$f'(b) = \lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$

## Link between differentiability and continuity

- If f(x) is differentiable at x = a, then f(x) is also continuous at x = a.
- The converse is not necessarily true!

Example: Show that y = |x| is differentiable for all  $x \neq 0$ .

### Higher-order derivatives

Now that we can take the derivative of a function, we can take the derivative of the derivative, and so on...

Notation	Name
$\overline{ \begin{array}{c} f'(x) \\ f''(x) = \frac{d}{dx} f'(x) \end{array} }$	1st-order derivative 2nd-order derivative
$\overset{\dots}{f^{(n)}(x)} = \tfrac{d}{dx} f^{(n-1)}(x)$	${n}$ th-order derivative

Derivative of a polynomial: basic rules

0

2

$$\frac{d}{dx}c = 0$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad \text{for } n \neq 0$$

Derivative of a polynomial: basic rules

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4

 $\frac{d}{dx}(cf(x))=c\frac{df}{dx}$ 

$$\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

### Examples

Compute the derivative of  $f(x) = 3x^2 + 7x - 5$ .

### Derivative of the exponential/logarithm

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

For a different base:

$$\frac{d}{dx}a^{x} = a^{x}\ln a$$
$$\frac{d}{dx}\log_{a} x = \frac{1}{x\ln a}$$

- The last two rules will follow from the chain rule
- Do not confuse the rules for  $a^x$  and  $x^n$ !

### Product and quotient rules

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

Examples: Compute

• 
$$\frac{d}{dx}(x^2e^x)$$
  
•  $\frac{d}{dx}\frac{x+1}{x+3}$ 

## Derivatives of trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

#### Examples

Using the rules from the previous slide, show that

$$\frac{d}{dx}\csc x = -\cot x \csc x$$
$$\frac{d}{dx}\sec x = \tan x \sec x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

Example solution:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{1'\cdot\sin x - \sin'(x)\cdot 1}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}.$$

(This will be easier once we cover the chain rule)

### Derivatives of inverse trigonometric functions

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

These rules can be derived by means of *implicit differentation*, which we will cover later.

## The chain rule

- Useful for composite functions  $F = f \circ g$  ("f after g")
- If f(x) and g(x) are differentiable, then the composite function F(x) is also differentiable and

 $F'(x) = f'(g(x)) \cdot g'(x).$ 

Example

Compute 
$$\frac{d}{dx}e^{\sqrt{\cos x}}$$
.

# Example

Compute 
$$\frac{d}{dx}x^{\sin(x)}$$
.