# Complex Numbers Introduction to Engineering Mathematics

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#### Overview

- Motivation
- Number systems
- Graphical representation of complex numbers
- Modules, argument, complex conjugate
- Complex arithmetic

Motivation: solving quadratic equations

Find x so that  $x^2 - 2x + 2 = 0$ .

- As D = -4 < 0, there are **no real solutions**
- If we set  $i = \sqrt{-1}$ , then we find **two solutions**.

OK to calculate with i, as long as we remember

$$i^2 = -1.$$

#### Number systems

Symbol	Elements	Used for
N Z Q R	0, 1, 2, , -1, 0, 1, 2, Fractions $n/m$ $\mathbb Q$ and irrational numbers: $e, \pi$ ,	Counting Adding/subtracting Dividing Limits
C	$\overset{\cdot\cdot\cdot}{a}+ib$ , with $a,b\in\mathbb{R}$	Solving equations

# Graphical representation of complex numbers

If z = x + iy is a complex number, then

- $\operatorname{Re}(z) = x$  (the real part)
- Im(z) = y (the imaginary part)

are both real numbers and  $\left( x,y\right)$  determines a point in the plane.



Argand plane:

- x-axis: Real axis
- y-axis: Imaginary axis

## Example

In the complex plane, find the location of:

$$\begin{array}{l} \bullet \ z_1 = 1 \\ \bullet \ z_2 = 2 + 3i \\ \bullet \ z_3 = -2i \\ \bullet \ S = \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 1\} \end{array}$$

## Modulus and argument

• **Modulus** (absolute value) of *z*: distance to the origin.

$$|z|=d(z,O)=\sqrt{x^2+y^2}.$$

• Argument of *z*: angle with positive *x*-axis.

$$\operatorname{Arg}(z) = \theta \in (-\pi, \pi]$$
 if

$$\tan \theta = \frac{y}{x}.$$



#### Polar representation of complex numbers

If r is the modulus and  $\theta$  the argument of z = x + iy, then

$$\begin{cases} x = r\cos\theta\\ y = r\sin\theta \end{cases}$$

This gives us the **polar representation of** *z*:

$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$

# Example

Find the polar representation of

• 
$$z_1 = i$$
  
•  $z_2 = 1 + i$   
•  $z_3 = -\sqrt{3} - i$ 

## Complex conjugate

If z = x + iy, then the complex conjugate  $\bar{z}$  is given by

$$\bar{z} = x - iy.$$

Properties:

$$\begin{array}{l} \mathbf{0} \ \ \mathrm{Re}(\bar{z}) = \mathrm{Re}(z) \\ \mathbf{2} \ \ \mathrm{Im}(\bar{z}) = - \mathrm{Im}(z) \\ \mathbf{3} \ \ |\bar{z}| = |z| \\ \mathbf{4} \ \ \mathrm{Arg}(\bar{z}) = - \mathrm{Arg}(z) \end{array}$$



## Adding and subtracting complex numbers

Complex numbers can be added/subtracted component-wise: if z = x + iy and w = a + ib, then

$$z \pm w = (x + iy) \pm (a + ib)$$
$$= (x \pm a) + i(y \pm b)$$

This has a nice geometric interpretation via the **parallellogram rule**:

- Draw a parallellogram with sides z and w
- z + w is at the end of the diagonal



## Multiplying complex numbers

If 
$$z = x + iy$$
 and  $w = a + ib$ , then (using  $i^2 = -1$ )

$$zw = (x + iy) \cdot (a + ib)$$
$$= (xa - yb) + i(ya + xb)$$

Properties:

1 
$$z\bar{z} = |z|^2$$
  
2  $\overline{zw} = \bar{z} \cdot \bar{w}$ 

# Product of complex numbers in polar form

Write

$$w = r(\cos \theta + i \sin \theta)$$
$$z = s(\cos \phi + i \sin \phi)$$

Then we get the following nice form for the complex product:

$$wz = \mathop{rs}_{|wz|} (\cos(\underbrace{\theta+\phi}_{\operatorname{Arg}(wz)}) + i\sin(\theta+\phi))$$

In particular, we get

• 
$$|wz| = rs = |w||z|$$
  
•  $\operatorname{Arg}(wz) = \theta + \phi = \operatorname{Arg}(w) + \operatorname{Arg}(z)$ 

#### De Moivre's theorem

From the product rule, we get

$$\begin{split} |z_1z_2\cdots z_n| &= |z_1||z_2|\cdots |z_n|\\ \mathrm{Arg}(z_1z_2\cdots z_n) &= \mathrm{Arg}(z_1)+\cdots + \mathrm{Arg}(z_n). \end{split}$$

Substituting  $z_1 = \ldots = z_n = \cos \theta + i \sin \theta$  gives us De Moivre's theorem:

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta).$$

## Division of complex numbers

We put

$$\frac{z}{w} = \frac{x + iy}{a + ib}.$$

- How can we make sense of this complex number?
- Multiply by the conjugate:

$$\frac{z}{w} = \frac{x+iy}{a+ib}\frac{a-ib}{a-ib} = \frac{ax+by}{a^2+b^2} + i\frac{ay-bx}{a^2+b^2}.$$

Properties:

 $|z/w| = |z|/|w| \quad \text{and} \quad \operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w).$ 

## Useful properties of complex numbers

• 
$$\overline{z+w} = \overline{z} + \overline{w}$$

• 
$$\overline{zw} = \overline{z}\overline{w}$$

• 
$$\bar{z} = z$$

• 
$$zw = 0$$
 iff  $z = 0$  or  $w = 0$ 

Property: to take the conjugate of a complicated expression, it suffices to take the conjugate of every term.

Example: Given 
$$z = i \frac{Z-1}{Z+1}$$
, compute  $\bar{z}$ .

# Examples

- Find the modulus and argument of z = (3 + 5i)(4 2i).
- Simplify the complex number  $z = \frac{7+3i}{4i}$ .

## Examples

Simplify the following complex numbers as much as possible:

• 
$$z = \frac{1+i}{1-i}$$
  
•  $z = i^{2022}$ 

#### Caveat

Keep in mind that, for complex numbers,

 $\sqrt{ab} \neq \sqrt{a}\sqrt{b}.$ 



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Source: Tom Gauld