# Theory of equations (2/2): Polynomial equations Introduction to Engineering Mathematics

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#### Overview

- Every polynomial of degree N has N roots
  - Some of these roots may be *complex* (e.g.  $x^2 + 1$ )
  - Some of these roots may be the same (e.g.  $x^2 + 2x + 1$ )
- Roots correspond to factors of the polynomial
- There is no algorithm for finding all roots of a polynomial
- If a real polynomial has a complex root z, then the complex conjugate  $\bar{z}$  is also a root (e.g.  $x^3-x^2+x-1)$

## Recall

Polynomial of degree n:

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

- The number n is called the **degree** of P(x).
- A root or zero is a number  $\alpha$  such that  $P(\alpha) = 0$ .
- Roots can be real  $(\alpha \in \mathbb{R})$  or complex  $(\alpha \in \mathbb{C})$ .
- A factor is a polynomial F(x) such that P(x) = F(x)Q(x) for some other polynomial Q(x).
  - Linear factor:  $F(x) = x \alpha$
  - Quadratic factor:  $F(x) = Ax^2 + Bx + C$

## Remainder theorem (special case)

If  $P(\boldsymbol{x})$  is a polynomial, then P(h) is the remainder of  $P(\boldsymbol{x})$  divided by  $\boldsymbol{x}-h.$ 

## Corollary

Note: "Corollary" means "consequence".

If P(x) is a polynomial with zero  $\alpha \in \mathbb{C}$  (in other words,  $P(\alpha) = 0$ ), then  $x - \alpha$  is a factor of P(x):

$$P(x) = (x - \alpha)Q(x).$$

#### Example

Find all the factors of  $P(\boldsymbol{x})=2\boldsymbol{x}^3+3\boldsymbol{x}^2-1.$ 

### Remainder theorem (general version)

If 
$$P(x)$$
 is a polynomial with  $distinct \ {\rm zeros} \ \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{C}$ , then  $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$  is a factor of  $P(x)$ :

$$P(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)Q(x).$$

Notes:

• There are at most n distinct zeros, where n is the degree of P(x) (see later).

### Examples

Find a polynomial of degree 4 with roots  $\pm i,$   $\pm 2,$  and such that P(3)=25.

#### Examples

Find a polynomial of degree 4 with roots 0 and -2, and where the root -2 has multiplicity 3.

#### How many roots can a polynomial have?

**Theorem:** A polynomial  $P(x) \neq 0$  cannot have more than n distinct roots, where  $n = \deg P(x)$ .

**Proof:** Assume that there are m distinct roots  $\alpha_1,\ldots,\alpha_m$  , with  $m>\deg P(x).$  Then by the remainder theorem,

$$P(x)=(x-\alpha_1)\cdots(x-\alpha_m)Q(x).$$

The left-hand side has degree n, whereas the right-hand side has degree at least m > n. This is a contradiction.

#### Relation between roots and coefficients

Define the symmetric polynomials:

$$\begin{array}{l} \bullet \ S_1 = a_1 + \dots + a_n \\ \bullet \ S_2 = a_1 a_2 + a_1 a_3 + \dots + a_1 a_n + a_2 a_3 + \dots + a_{n-1} a_n \\ \bullet \ S_3 = a_1 a_2 a_3 + \dots + a_{n-2} a_{n-1} a_n \\ \bullet \ \dots \\ \bullet \ S_n = a_1 a_2 \dots a_n \end{array}$$

Then:

$$\begin{array}{l} (x-a_1)(x-a_2)\cdots(x-a_n)=\\ x^n-S_1x^{n-1}+S_2x^{n-2}-S_3x^{n-3}+\cdots+(-1)^nS_n. \ \ (1) \end{array}$$

#### Example

Given  $P(x) = x^3 + 2x^2 - 3x - 1$  with roots  $\alpha$ ,  $\beta$ , and  $\gamma$ , find the value of  $\alpha^2 + \beta^2 + \gamma^2$ .

## The fundamental theorem of algebra

**Theorem:** Each polynomial has *at least one* root (which may be complex).

Proof: Difficult.

**Consequence:** Each polynomial of degree n has exactly n roots (which may be same).

## How to find roots?

- Degree 2: formula for quadratic equation
- Degree 3, 4: formulas exist, but they are very complicated
- Degree 5 and up: no general formula exists

In general, proceed via trial and error, or numerically.

# Example

Factorize 
$$P(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$$
.

#### Complex conjugates theorem

**Theorem:** If P(x) is a polynomial with real coefficients, then complex roots appear in *conjugates*.

In other words, if  $z = \alpha + i\beta$  is a root with multiplicity p, then  $\bar{z} = \alpha - i\beta$  is also a root with multiplicity p.