Integration: Partial Fractions Introduction to Engineering Mathematics

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What are partial fractions?

Every rational function can be written as a sum of "simple" partial fractions. For example

$$\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}.$$

In this lecture, we will find a recipe for the coefficients and terms in the partial fraction expansion.

Why are partial fractions useful?

The advantage is that the partial fractions are *much* easier to integrate:

$$\int \frac{x+2}{x^3-x} dx = -2 \int \frac{dx}{x} + \frac{3}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1}$$
$$= -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

How to find the partial fraction expansion

Goal

To integrate any rational function: determine

$$\int \frac{P(x)}{Q(x)} dx = ???$$

where P(x) and Q(x) are polynomials.

Step 1: Divide if necessary

If the degree of P(x) is greater than or equal to the degree of Q(x), do a polynomial division:

$$\int \frac{P(x)}{Q(x)} dx = \int S(x) dx + \int \frac{R(x)}{Q(x)} dx,$$

with S(x) the quotient and R(x) the remainder.

- Recall: $\deg R(x) < \deg Q(x)$.
- From now on, we will suppose that this division has already been done, so that $\deg P(x) < \deg Q(x)$.

Special case 1: Linear denominator

If $Q(\boldsymbol{x})$ is a linear polynomial, i.e. $Q(\boldsymbol{x}) = A\boldsymbol{x} + B$, then our integral takes the form

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{K}{Ax+B} dx$$
$$= \cdots$$

Special case 2: Quadratic denominator

If Q(x) is a quadratic polynomial, then several cases are possible. After completing the square, we can have one of the following forms:

If deg P(x) = 0: • $\int \frac{dx}{x^2 - a^2}$ • $\int \frac{dx}{x^2 + a^2}$ If $\deg P(x) = 1$:

•
$$\int \frac{x \, dx}{x^2 - a^2}$$

•
$$\int \frac{x \, dx}{x^2 + a^2}$$

Step 2: Find the roots of the denominator

Do a factorization of the denominator $Q(\boldsymbol{x})$ into factors with \mbox{real} coefficients.

You will find:

- Some linear factors $(x \alpha)$, with α roots of Q(x)
- Some quadratic factors (Ax^2+Bx+C) that cannot be further reduced.

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Important: Do not split into complex factors. For example:

$$x^3 + x = x(x^2 + 1).$$

Stop here, don't factor into x(x+i)(x-i).

Case 2.1: Distinct roots

Assume that $Q(x)=(x-\alpha_1)\cdots(x-\alpha_k)\text{, with all }\alpha_i\text{ distinct}$ and real.

Then the partial fraction expansion becomes

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \dots + \frac{A_k}{x-\alpha_k},$$

where the coefficients A_1,\ldots,A_k can be determined by adding the terms together and comparing with the left-hand side.

Find
$$\int \frac{x+2}{x^3-x} dx$$

Case 2.2: Irreducible quadratic factors

- For each quadratic factor, put a *linear term* in the numerator of the partial fraction.
- Deal with the linear factors as before.

$$\frac{x+2}{x^3+x} = \frac{x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \cdots$$

Therefore
$$\int \frac{x+2}{x^3+x} dx = \cdots$$

Case 2.3: Repeated linear factors

• If Q(x) has a repeated factor $(x - \alpha)^p$, then add p terms to the partial fraction expansion:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \dots + \frac{A_p}{(x-\alpha)^p} + [\text{other PFE}]$$

• Deal with other linear and quadratic factors as before.

Determine the partial fraction expansion of $\frac{1}{x^2(x-1)^3}$.

Find
$$\int \frac{dx}{x^3 - 5x^2 + 8x - 4}.$$

Summary

- 1 If $\deg P(x) \ge \deg Q(x)$, do polynomial division.
- **2** Factor the denominator Q(x) and write partial fractions for each root:
 - Distinct roots (roots with multiplicity 1):

$$\mathsf{PF} = \frac{A}{x - \alpha}$$

• Irreducible quadratic factors:

$$\mathsf{PF} = \frac{Ax+B}{x^2+\cdots}$$

Root with multiplicity p:

$$\frac{A_1}{x-\alpha} + \dots + \frac{A_p}{(x-\alpha)^p}$$

- Sind the coefficients in the partial fraction expansion by solving a system of equations.
- **4** Integrate the partial fraction expansion.